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Delayed Self-Organized Criticality and Earthquake Modeling

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Delayed self-organized criticality is defined. It is shown to preserve the power-law behavior of self-organized criticality with a significant change in the exponents. A delayed version of the Ito–Matsuzaki model for earthquakes is constructed and studied. This model explains some fractal features of earthquakes as well as the Gutenberg–Richter and Omori laws. Furthermore the b value obtained from the delayed model is closer to observations than the b value of the undelayed model.

1. INTRODUCTION

Geophysical information indicates that most of the great earthquakes occur on the same zones located around tectonic faults. To explain this, regard the earth's crust as a collection of a small number of very large tectonic plates moving at velocities of the order of a few centimeter per year. The boundaries between these plates form faults. Due to the inner motions of the earth, the plates press each other and restore the energy until reaching a critical value. Then, the tectonic plates undergo a sudden and very rapid motion, and the energy is dissipated through the faults: an earthquake occurs. Generally, this sudden motion changes the plates' energy. Aftershocks occur if the energy reaches its threshold again. In the same way aftershocks may be followed by other after-aftershocks and so on (Utsu, 1970).

Earthquakes have several fractal features (Turcotte, 1992). The frequency of earthquakes $N(>M)$ having magnitude greater than M is given by the following empirical relation (Gutenberg and Richter, 1954);

$$\log N(>M) = -bM + a \tag{1}$$

where the value of b ranges between 0.8 and 1.06 (Evernden, 1970). The

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relation between the area of the fractured zone, s, of an earthquake and its magnitude is (Turcotte, 1992)

$$\log s \propto M \tag{2}$$

Then,

$$N(>s) \propto s^{-b} \tag{3}$$

and,

$$n(s) \propto s^{-(b+1)} \tag{4}$$

where $n(s)$ is the frequency of earthquakes having size s.

Omori described the occurrence rate of aftershocks $R(t)$ as a power law of time t following the main shock which occurred at time t_0 as follows (Omori, 1894; Utsu, 1961):

$$R(t) \propto \frac{1}{(t + t_0)^p} \tag{5}$$

The concept of self-organized criticality (Bak *et al.*, 1987) is aimed to model self-sustaining systems with many degrees of freedom. An interactive dissipative system with many degrees of freedom is said to be in a state of self-organized criticality if it maintains itself near a critical point. Self-organized criticality has applications in various geophysical fields, e.g., earthquake dynamics (Ito and Matsuzaki, 1990; Matsuzaki and Takayasu, 1991), plate tectonic behavior (Sornette *et al.*, 1990), volcanic activity (Diodati *et al.*, 1991), etc.

Delayed models aim to reach the optimal description for systems in which the occurrence of any event is a result of a main event that occurred in the past. This concept applies in many real systems. In medicine, a patient may get sick, but symptoms may not appear until later. In economics, an old economic mistake may lead to the failure of a great country, as in the former USSR. In seismology, earthquakes result from crustal deformation occurring over many years.

2. SELF-ORGANIZED CRITICALITY AND EARTHQUAKES

The sandpile is the most famous example for self-organized criticality. Consider a 2-dimensional lattice with open boundaries; integer variables $z(i, j)$ represent the height of the sandpile at the site (i, j). Initially each site contains a random number of grains between 0 and the critical value (set equal to 3). Sand is added to the lattice by the following procedure:

Step 1: A sand grain is randomly added to a site.

Step 2:

$$\begin{cases} \text{If} & z(i,j) > 3 \\ \text{then} & z'(i,j) = z(i,j) - 4 \\ \text{and} & z'(i \pm 1, j \pm 1) = z(i \pm 1, j \pm 1) + 1 \end{cases} \tag{6}$$

Step 2 is repeated until all $z(i, j)$ become less than 4. Then the avalanche ends and we return to step 1. For each avalanche, we calculate its size s and the number of distinct sites s_d.

Dropping 5000 sand grains on a 100 × 100 lattice, we find that the probability distributions of s and s_d obey power laws,

$$P(s) = s^{-1.08} \tag{7}$$

$$P(s_d) = s_d^{-1.18} \tag{8}$$

Our results are very close to those obtained in Bak and Creutz (1994).

Now, we describe the earthquake occurrence in cellular automaton language. Consider a 2-dimensional lattice where sites are considered as tectonic plates. Adding sand acts like the inner motion of the earth. The variables $z(i, j)$ correspond to the restored energy. An avalanche represents an earthquake. After an earthquake is completed, the heights of all sites which slipped during the earthquake either increase or decrease by 1 with equal probabilities. Aftershocks originate at those sites whose heights exceeds the threshold. The obtained aftershocks obey Omori's law (Ito and Matsuzaki, 1990).

We dropped 4000 sand grains on a 100 × 100 cellular automaton model and calculated the size of each earthquake. We found that the size–frequency relation obeys the power law

$$n(s) \propto s^{-1.74} \tag{9}$$

Comparison between (4) and (9) gives $b \approx 0.74$, which is very close to that estimated by Ito and Matsuzaki (1990) and agrees with observations (Evernden, 1970). So, self-organized criticality is the best approach for earthquake modeling.

3. DELAYED SELF-ORGANIZED CRITICALITY

In the preceding models, the event which will occur at time $t + 1$ depends only on the state of the model at time t. But it is known that in most real systems the behavior of the system at time $t - 1$ affects that at time $t + 1$. Here we introduce the dependence of an event at time $t + 1$ on events at both times t and $t - 1$. This is what we mean by the word "delay." Of course, this definition may be generalized, but this is deferred to future work.

First, we constructed a delayed sandpile using the following procedure:

Step 1: Initially, we begin with two distinct lattices with heights $z_1(i, j)$ and $z_2(i, j)$ chosen randomly between 0 and 3.

Step 2:

$$z = [z_1(i, j) + z_2(i, j)]/2$$

If $\mathrm{Int}(z) < z$ and RND < 0.5, then $z = z + 1$, where RND is a uniformly distributed random number. We have

$$z_3 = \mathrm{Int}(z) \tag{10}$$

Add a sand grain randomly to the lattice with heights z_3.

Step 3:

$$\begin{cases} \text{If} & z_3(i, j) > 3 \\ \text{then} & z_3'(i, j) = z_3(i, j) - 4 \\ \text{and} & z_3'(i \pm 1, j \pm 1) = z_3(i \pm 1, j \pm 1) + 1 \end{cases} \tag{11}$$

This step is repeated until all $z_3(i, j)$ become less than 4.

Step 4:

$$\begin{cases} z_1(i, j) = z_2(i, j) \\ z_2(i, j) = z_3(i, j) \end{cases} \tag{12}$$

For each avalanche we calculate its size s and the number of distinct s_d. The probability distributions of s and s_d obey also power laws,

$$P(s) = s^{-1.5} \tag{13}$$

$$P(s_d) = s_d^{-1.59} \tag{14}$$

Therefore, delay preserves the power-law behavior of self-organized criticality, but with a significant change in the exponents.

Second, we present a delayed version of the Ito–Matsuzaki model for earthquakes as follows,

Step 1: We have two distinct lattices with heights $z_1(i, j)$ and $z_2(i, j)$ chosen randomly between 0 and 3.

Step 2:

$$z = [z_1(i, j) + z_2(i, j)]/2$$

If $\mathrm{Int}(z) < z$ and RND < 0.5, then $z = z + 1$, where RND is a uniformly distributed random number. We have

$$z_3 = \mathrm{Int}(z) \tag{15}$$

Add one grain randomly to $z_3(l_1, l_2)$, where l_1 and l_2 are randomly chosen.

Step 3:

$$\begin{cases} \text{If} & z_3(i,j) > 3 \\ \text{then} & z_3'(i,j) = z_3(i,j) - 4 \\ \text{and} & z_3'(i \pm 1, j \pm 1) = z_3(i \pm 1, j \pm 1) + 1 \end{cases} \tag{16}$$

This step is repeated until all $z_3(i, j)$ become less than 4.

Step 4: Perturb the value z_3 of all sites which slipped in step 3 by either increasing or decreasing by 1 with equal probabilities. If there exist critical sites, then return to step 3.

Step 5:

$$\begin{cases} z_1(i,j) = z_2(i,j) \\ z_2(i,j) = z_3(i,j) \end{cases} \tag{17}$$

Return to step 2.

The obtained aftershocks satisfy Omori's law. The size–frequency relation obeys the power law

$$n(s) \propto s^{-1.98} \tag{18}$$

The estimated $b \approx 0.98$ also agrees with observations (Evernden, 1970).

Therefore, delayed self-organized criticality models for earthquakes preserve the attractive feature of fractality, and agree with the Omori and Gutenberg–Richter laws, in addition to being closer to real systems, where delay is a basic feature. The estimated b value is closer to observations than that estimated by the ordinary Ito–Matsuzaki model.

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